

Topological stability of broken symmetry on fuzzy spheres

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Abstract

We study the spontaneous symmetry breaking of $O(3)$ scalar field on a fuzzy sphere S_F^2 . We find that the fluctuations in the background of topological configurations are finite. This is in contrast to the fluctuations around a uniform configuration which diverge, due to Mermin-Wagner-Hohenberg-Coleman theorem, leading to the decay of the condensate. Interesting implications of enhanced topological stability of the configurations are pointed out.

1 Introduction

Field theories on non-commutative geometries are inherently non-local[1, 2, 3, 4, 5]. This feature gives rise to novel behaviour such as the mixing of infrared and ultraviolet scales. These lead to new ground states with spatially varying condensates[6, 7, 8, 9]. Many non-perturbative studies have established that fuzzy spaces such as Groenewold-Moyal plane, fuzzy spheres allow for the formation of stable non-uniform condensates as ground states[10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. Nonlocality plays an essential role in all these. Exploring implications of this nonlocal nature of field theories is one of the intensive research activity in the field of noncommutative physics today.

In the past, we have analysed the spontaneous breakdown of global symmetries and the corresponding behaviour of Goldstone modes through simulations[17, 18]. The main aim of this paper is to study the interplay of SSB and topological features of the sigma models on fuzzy spheres. This issue is important because the Mermin-Wagner-Hohenberg-Coleman (MWHC) theorem states that there can be no SSB of continuous symmetry on 2-dimensional commutative spaces[20, 21, 22, 23]. This theorem was established rigorously by summing up all the infrared contributions in the context of field theories with local interactions. The resulting diverging fluctuations of the Goldstone modes prevent the symmetry from being spontaneously broken. No SSB also implies no magnetisation. There is

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no obvious generalisation of the MWHC theorem for the non-commutative spaces, since the theorem relies strongly on the locality of interactions. Gubser and Sondhi, [6], have shown that the Goldstone mode fluctuations become severe in the non-commutative case. These authors infer, after analysing the fluctuations in the background of a uniform condensate, that a condensate with only zero momentum/angular momentum mode are unstable. However non-commutative spaces admit non-uniform solutions and one can ask the question what happens to the stability of these configurations. Non-uniform condensates naturally have an infra-red cut-off for the fluctuations. This cut-off can soften the otherwise divergent contributions of the Goldstone modes. It would be desirable to compute the fluctuations by analytical methods as was done for the MWHC theorem. Only few models allow for exact treatment of SSB even in the commutative spacetime. It is difficult to analytically sum the fluctuations in the present case. But in the absence of such a study numerical simulations can provide answers in the present context.

In a previous work of ours, we explored these issues for $U(1)$ symmetry using numerical simulations [18]. As expected, we found that the uniform condensates are unstable. But the condensates with higher angular momentum components still survived the fluctuations. However when the thermodynamic and continuum limits were taken, the fluctuations in the momentum mode above the cut-off increased. The Goldstone modes were capable of destroying the low lying modes of the non-uniform condensate.

In this paper we analyse the $O(3)$ symmetry on fuzzy sphere. Nonlinear sigma models on fuzzy spheres have been studied in the literature exploring topological features [24, 25]. We expect that if we repeat the calculations of $U(1)$ for this case the results will be more or less similar. But the $O(3)$ case is significantly different from our previous study in an important way. In this case there are non-uniform configurations which are topologically stable. For example the hedgehog type configuration. A condensate which is topologically stable can be unstable against fluctuations with energies of the order of the system size. So we expect that Goldstone modes whose energy goes as logarithm of the system size may not be able to destroy the hedgehog condensates. To settle this, we calculate the fluctuations around such configurations via numerical simulations. Our numerical results show, that indeed, topological stability and nonlocal interactions make the lowest non-zero mode of the condensate stable.

This paper is organised as follows: In Sec.2 we describe the model and the non-trivial topological configurations. The numerical simulations and the results are discussed in Sec.3. In Sec.4 we provide our conclusions.

2 $O(3)$ Model and Topological Condensates on S_F^2

The fuzzy sphere is described by the coordinates X_i satisfying the following $SU(2)$ algebra,

$$[X_i, X_j] = \frac{i\alpha\epsilon_{ij}^k X_k}{\sqrt{j(j+1)}}, \quad \sum_i X_i^2 = R^2. \quad (1)$$

Here R is the radius of the fuzzy sphere. One can choose the angular momentum $j = \frac{N}{2}$ representation for the fuzzy sphere.

With this choice the most general action for $O(3)$ hermitian scalar field Φ_i ($i = 1, 2, 3$) upto quartic interactions takes the form,

$$S(\Phi) = \frac{4\pi}{N} \text{Tr} \left[\sum_i |L_i, \Phi|^2 + R^2 (r|\Phi|^2 + i\beta\epsilon_{ijk}\Phi_i\Phi_j\Phi_k + \lambda(|\Phi|^2)^2 + \mu|[\Phi_i, \Phi_j]|^2) \right] \quad (2)$$

In the mean-field the above theory admits a uniform condensate for $r < 0$. However as discussed above the fluctuations of the Goldstone mode render this solution unstable. Apart from the uniform condensate the above model admits many meta-stable solutions. To simplify our arguments we first consider the case $\beta = 0, \mu = 0$:

The number of meta stable configurations increases with the matrix size of Φ . Among them we are interested on those which are stable due to topological obstructions. For example,

$$\Phi_i = \alpha L_i, \quad \text{with} \quad \alpha = \sqrt{\frac{\frac{2|r|}{\lambda}}{N^2 - 1}} \quad (3)$$

The analog of this configuration in continuum space is the hedgehog configuration where the $O(3)$ spin vector on the sphere is pointing radially outward. The spin is parallel to the position vector on the sphere. This configuration is topologically stable as it cannot be smoothly deformed to a uniform one. Similarly the above configuration cannot be smoothly deformed to $\Phi = I$ which is also a solution. The above configuration corresponds to a winding one map from the physical space S_F^2 to the vacuum manifold which is S_F^2 . All topologically stable configurations in the continuum limit, can be characterised by the second homotopy group $\Pi_2(S^2)$. For a discussion on topological classification of the maps $S_F^2 \rightarrow S_F^2$ see [24, 25].

When we include the other terms in the action we have more parametric choices for the topological configurations. For example when $\beta \neq 0$, but $\mu = 0$ we have

$$\Phi_i = \alpha L_i, \quad \text{with} \quad \alpha = \frac{3\beta \pm \sqrt{9\beta^2 - 32r\lambda L^2}}{8\lambda L^2} \quad (4)$$

Then Eq.(4) shows clearly the possible existence of topological vacuum configurations even when $r \geq 0$. For example, when $r = 0$ we have

$$\Phi_i = \alpha L_i, \quad \text{with} \quad \alpha = \frac{3\beta}{\lambda (N^2 - 1)} \quad (5)$$

While it is important to study the topological stability in such a general situation the results are not characteristically different from $\beta = 0$. Here we focus on $\beta = 0$ in this paper for simulations but discuss general cases.

To study the net effect of topological nature of the background configuration and non-locality on fluctuations, we consider only the winding number one configuration which is given in Eq.(3). As mentioned our plan is to compute these fluctuations numerically. Even before computing the fluctuations one can make some general remarks about the behaviour of the fluctuations [18]. The effect of nonlocality basically provides a non-zero mass $O(\frac{\alpha}{N})$ to the Goldstone mode fluctuations. This puts an infrared cut-off for the fluctuations. From our previous study [18] it seems that this mass/cutoff is mode dependent, as only

higher modes of the condensate survived the fluctuations. As we will see from our results the combined effect of topology and nonlocality, the infrared cut-off drastically reduce the contribution of the fluctuations. In the next section we describe our numerical simulations.

3 Numerical simulations

We use “pseudo-heat bath” updating method in our numerical simulations which is described in detail in our earlier papers [17, 18]. In our method, for each choice of parameters, we choose an initial configuration given by Eq. (3). Fluctuations around this configuration are then generated by the above updating method. Since this configuration is a variational solution to minimising the classical action, it will thermalise as we update/include the thermal fluctuations. Once the initial configuration is thermalised we compute the observable M . We make measurements after every 10 updates of the entire matrix. We also use over-relaxation to reduce the auto-correlation of the configurations generated in the Monte-Carlo history.

In a numerical simulation, the condensate will not maintain its exact form as in Eq. (3) along the Monte-Carlo history. The configuration can evolve into different random $SU(2)$ rotated configurations of Eq.(3) as we keep updating it. To overcome this, one needs to rotate the configuration at each step of the Monte-Carlo history so that the configuration takes the form of Eq. (3). But this is a difficult and time consuming task. On the other hand one can have an observable made of Φ_i 's which is invariant under the $SU(2)$ rotations, e.g basis independent. For this purpose we define the following observable,

$$A_{ij} = \frac{1}{N^2} Tr(L_i \Phi_j), M = \sqrt{A^\dagger A} \quad (6)$$

M projects out the $l = 1$ angular momentum mode. Note that the initial configuration in Eq. (3) projects out only the $l = 1$ mode. Analysing the statistical behavior of M will give us a definite conclusion about the stability of the initial configuration. We mention here that $Tr(\sum_i \Phi_i^2)$ is also an $SU(2)$ invariant. But the information on the amplitudes of

different l modes gets lost in this form. Also comparatively the observable M may serve as an order parameter in the case of any phase transition of the hedgehog configuration to $\Phi_i = 0$ at high temperatures. We mention here that $l = 1$ is the lowest possible stable mode, as $l = 0$ mode will be unstable. One can consider configurations with higher winding, instead of Eq.(3), however we expect them to be more stable than the $l = 1$ condensate. This is because the infrared cut off will rise with higher winding configurations.

For practical reasons, the size of the matrix N , in other words size of the resolution scale is finite. So there are usually finite volume (R, N) effects. So a non-vanishing condensate Φ_i does not mean there is SSB. One needs to define suitable observable dependent on Φ_i which should scale with (R, N) appropriately in the thermodynamic limit ($N \rightarrow \infty, R \rightarrow \infty$) to conclude anything. Now there are two possible thermodynamic limits. If in the thermodynamic limit the ratio $\frac{R^2}{N}$ does not vanish then the space is described by a non-commutative algebra. This limit is of interest to us, as we expect that the MWHC theorem will hold good in the commutative thermodynamic limit.

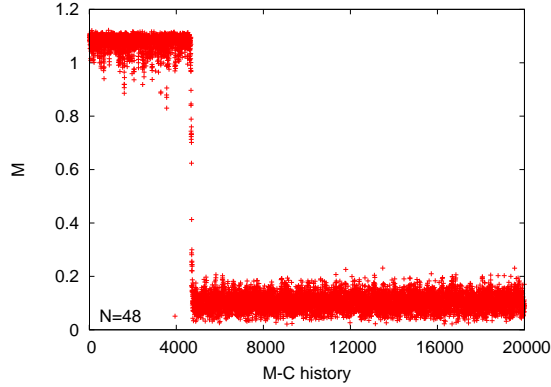


Figure 1: M vs Monte Carlo history for $N = 48$

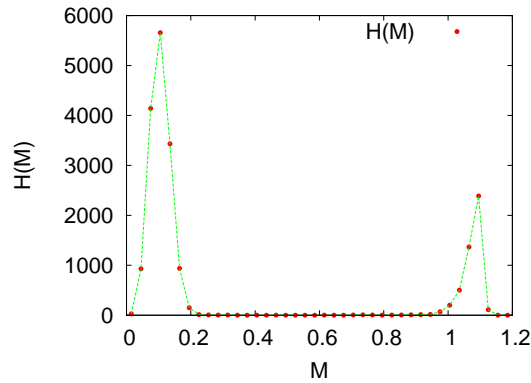


Figure 2: Histogram $H(M)$ for $N = 48$

4 Results and discussions

In our calculations we fix $\frac{R^2}{N} = 10$, $r = -8$. For simplicity we take $\lambda_1 = 0.25$. With this choice of parameters we do our simulations for five different sizes of the Φ matrices, $N = 48, 56, 64, 78, 96$. FIG.1 gives a typical Monte-Carlo history of our simulations for $N = 48$.

In FIG.1 M fluctuates around a value close to the initial value. Then M suddenly jumps to a small value and settles down. A histogram $H(M)$ of M clearly shows two peaks M as seen in FIG.2. The peak on the left has large $l = 0$ and small $l = 1$ component. The peak at higher value of M has large $l = 1$ component and small $l = 0$. This peak is close to the value of the initial configuration. So in this state fluctuations modify the initial configuration slightly and retain its topological nature. In our Monte-Carlo history we observed the $l = 1$ state decaying to $l = 0$ state but not vice-versa. This implies that due to finite volume effects, the uniform condensate is more stable than our initial hedgehog configuration for this case of $N = 48$.

To study the stability of the $l = 1$ configuration we considered both the commutative and non-commutative limit. For the commutative limit we fixed R^2 and considered higher values of N . We did not observe any change in the distribution of M in the $l = 1$ state.

The average value, and the fluctuations of M remain almost the same as we go from $N = 48 \rightarrow 64$, as can be seen in FIG.3. As for $N = 48$ the $l = 1$ configuration also decays for $N = 64$. This result suggests that the $l = 1$ topological configuration is not stable in the commutative continuum limit as expected.

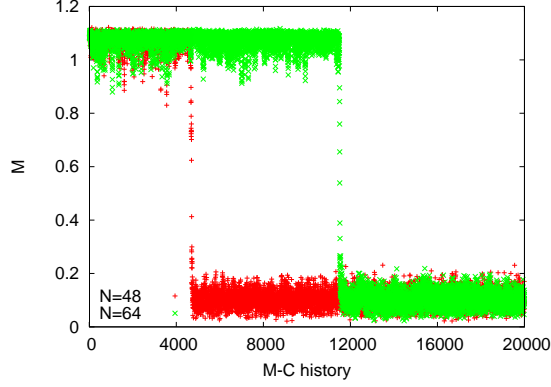


Figure 3: M-C history for fixed R^2

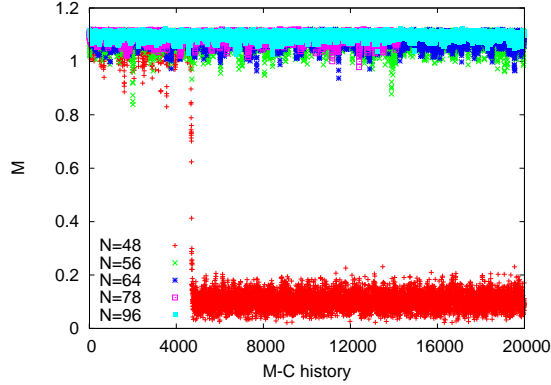


Figure 4: M-C history for fixed R^2/N

There is a complete change in the behavior as we consider the non-commutative limit, i.e fixed $\frac{R^2}{N}$ as we increase N . Except for the lowest $N = 48$ the $l = 1$ state did not decay during the entire run for higher N . In FIG.4 we show the Monte-Carlo history of $N = 48, 64, 96$. Unlike the commutative limit, the fluctuations of M decrease with N . This can be clearly seen in FIG.4. In FIG.5, we give the average value of M as a function of N . The average value of M increases slightly with N , with the variation decreasing with N . This suggests M will reach a finite value in the continuum limit. We also compute the fluctuations of M to see any possible scaling with the cut-off N . In FIG.6, we show $\chi = \langle M^2 \rangle - \langle M \rangle^2$ in the $l = 1$ state. The solid curve represents a fit, $f(N) \sim N^\alpha$ with $\alpha \sim -4$. This clearly suggest that the $l = 1$ state is stable in the $N \rightarrow \infty$ leading to spontaneous breaking of the $O(3)$ symmetry.

We also mention here that one can start with an initial uniform $l = 0$ configuration and consider fluctuations. We expect that the results be similar to that in ref.[18]. In ref.[18]

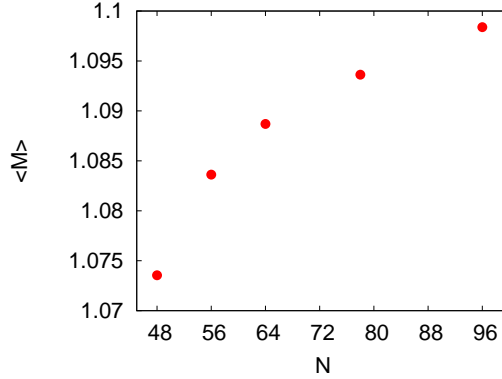


Figure 5: $\langle M \rangle$ vs N

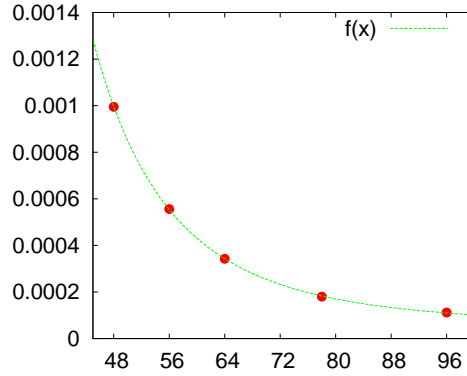


Figure 6: χ vs N

it was found that only the highest mode $l = \frac{N-1}{2}$ condenses. The fact that we find the $l = 1$ mode stable clearly shows that the topological nature of the initial configuration complements the effect of non-locality. These two effects drastically reduce the fluctuations.

5 Conclusions

In this analysis we have shown that topologically non-trivial configurations on the fuzzy sphere avoid the MWHC theorem much more dramatically than the non-topological symmetry breaking. The mass gap or the infrared cut-off in this case is large enough to render the fluctuations of the Goldstone modes finite. On the other hand for non-topological condensates the Goldstone modes are large enough to destroy almost all the modes except the few highest modes. It is interesting to scale λ and r as we go to continuum limit. The limit of noncommutative continuum geometry is always expected to maintain SSB and stable solitons as there is obstruction to MWHC theorem from nonlocality. This analysis will be presented elsewhere.

We have presented the simulations wherein the cubic Chern-Simons (CS) term is absent in the action Eq (2). The Chern-Simons term allows topological solitons even when the

quadratic mass term is positive upto some value. Interestingly with CS term the configuration $\phi_i = \alpha L_i$ is preferred over symmetric solution. On the otherhand, it is not expected to alter the picture of topological stability of the solutions. This term plays an important role in the emergent geometry in NC fuzzy spaces [26, 27]. What we find here in the simulations is that even in the absence of CS term, emergent fuzzy spaces can be stable. The stability of higher dimensional fuzzy spaces like CP_F^2 are of significance in this context [28]. The implications of this stability for extra-dimensional fuzzy spaces will be considered later.

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